A MATHEMATICAL NOTE

Gaussians of square cross-section

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Introduction. The optical/electronic beams used to draw microcircuits typically have circular cross-sections, and produce spots with Gaussian density profiles. Sometime in mid-1999 Richard Crandall contacted me to see what I might have to say about the production of spots with square cross-sections and sharper edges. I had then an immediate off-the-top-of-the-head response, and I take advantage of this idle moment to write it down. It was my impression that the idea which met Richard's specific need would have to fit naturally into the framework of a "generalized diffusion theory" to which he had several times alluded, but (on grounds that it was proprietary) had never described in detail. Richard found my idea "interesting," and at intervals of ten or fifteen minutes sent me several reports of its discovered implications, but the exchange was over within an hour of the time it began, and I have no idea whether edible fruit was ever harvested from this little bush.

My idea descends from the "super-ellipse" which Piet Hein (physicist-poet and friend of Bohr) devised originally to solve a traffic-congestion problem in Copenhagen, some amusing properties of which were once described by Martin Gardner. Hein's observation, and the essence of the present note, is displayed in Figure 1.

Square Gaussians. The functions in question have the form

$$g(x;n) \equiv \frac{\exp[-x^n]}{2\Gamma(\frac{n+1}{n})} \quad : \quad n = 2, 4, 6, \ldots$$

illustrated in Figure 2. *Mathematica* informs us that

$$\int_{-\infty}^{+\infty} g(x;n)\,dx = 1$$

At $n = 2$ we use $2\Gamma(\frac{3}{2}) = \sqrt{\pi}$ to recover Gauss' gaussian. The figure indicates that $\lim_{n\uparrow\infty} g(0;n) = \frac{1}{2}$, which follows analytically from $\Gamma(1) = 1$.

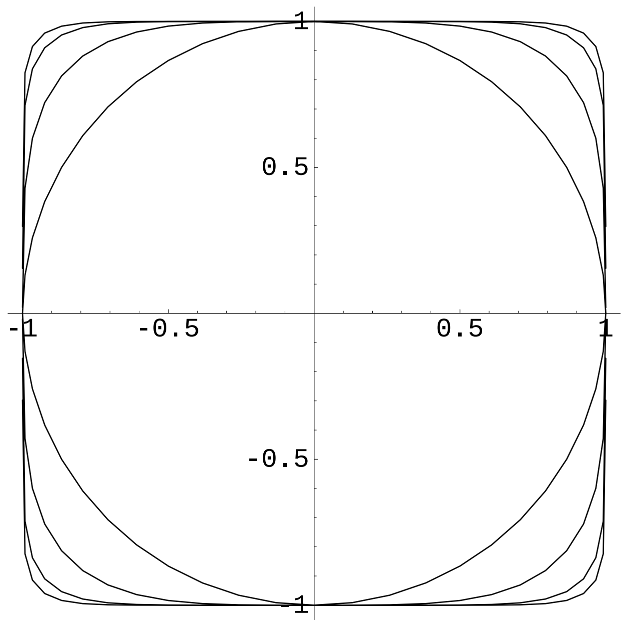

FIGURE 1: *Graphs of $x^n + y^n = 1$ with $n = 2, 4, 8, 12$. It was Piet Hein's observation that the circle becomes progressively more square as n advances through the even integers. "Super-ellipses" result from writing*

$$|x/a|^p + |y/b|^p = 1 \quad : \quad p > 2$$

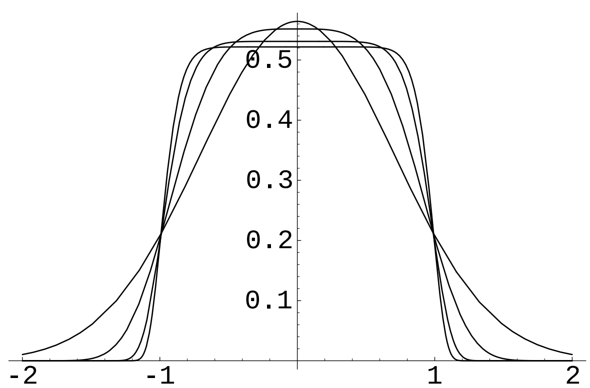

FIGURE 2: *Graphs of the normalized "square Gaussians" $g(x; n)$ with $n = 2, 4, 8, 12$.*

Looking to the second moment, we find

$$\langle x^2\rangle \equiv \int_{-\infty}^{+\infty} x^2 \cdot g(x;n)\,dx = \frac{\Gamma(\frac{3}{n})}{n\Gamma(\frac{n+1}{n})}$$
$$\downarrow$$
$$= \int_{-1}^{+1} x^2 \cdot \tfrac{1}{2}\,dx = \tfrac{1}{3} \text{ as } n \uparrow \infty$$

Mathematica answers $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{4}k^2\right)$ when asked for the Fourier transform of $g(x;2)$, but reports the transforms of $g(x;4)$, $g(x;6)$ ...to be hypergeometric, and of rapidly increasing complexity. This I read as indication that one cannot expect to obtain useful generalizations of the Hermite polynomials from $g(x;n)$.

One does, however, obtain tractable generalizations of the Gaussian representation of the δ-function:

$$\delta(x-a) = \lim_{\xi\downarrow 0} g(\tfrac{x-a}{\xi};n)$$

Figure 3 tells the story. The derivatives of $g(\frac{x-a}{\xi};n)$ are manageable, and supply representations of the derivatives of $\delta(x-a)$; see Figure 4.

Dimensional generalization is straightforward

$$g(x,y;n) = g(x;n)\cdot g(y;n)$$

and supplies the distributions (see Figure 5) which I recommended to Richard's attention.

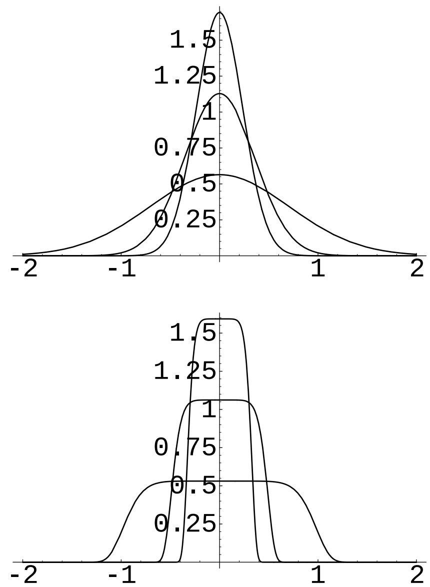

FIGURE 3: *The upper figure recalls the mechanism underlying the familiar Gaussian representation of the Dirac δ-function. In the lower figure $g(\frac{x-a}{\xi};8)$ has been pressed into the same service. In the limit $n\uparrow\infty$ we recover the "box function representation ."*

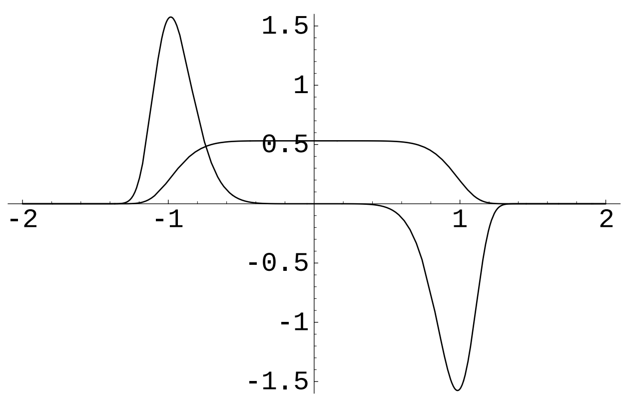

FIGURE 4: *Superimposed graphs of $g(x;8)$ and*

$$\tfrac{d}{dx}g(x;8) = -8x^7 g(x;8)$$

which supplies a representation of $\delta'(x-a)$.

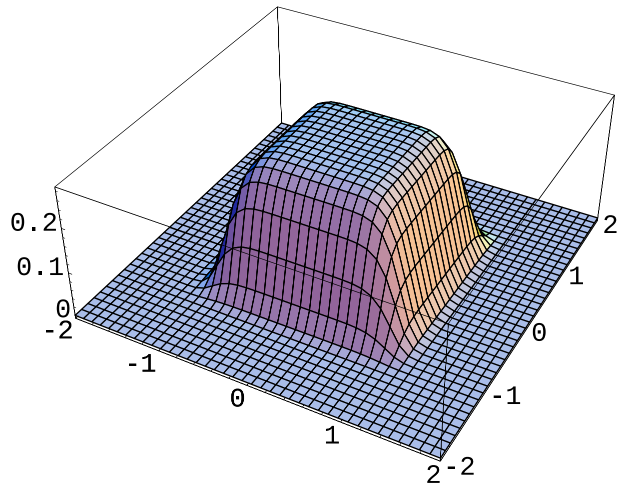

FIGURE 5: *Density profile of a beam of square cross-section. The function plotted is*

$$g(x, y; 8) = g(x; 8) \cdot g(y; 8)$$